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Exact Nonlinear Decomposition of Ideal-MHD Waves Using Eigenenergies

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Abstract

In this paper, we introduce a new method for exact decomposition of propagating, nonlinear magnetohydrodynamic (MHD) disturbances into their component eigenenergies associated with the familiar slow, Alfvén, and fast wave eigenmodes, and the entropy and field-divergence pseudoeigenmodes. First, the mathematical formalism is introduced, where it is illustrated how the ideal-MHD eigensystem can be used to construct a decomposition of the time variation of the total energy density into contributions from the eigenmodes. The decomposition method is then demonstrated by applying it to the output of three separate nonlinear MHD simulations. The analysis of the simulations confirms that the component wave modes of a composite wavefield are uniquely identified by the method. The slow, Alfvén, and fast energy densities are shown to evolve in exactly the way expected from comparison with known linear solutions and nonlinear properties, including processes such as mode conversion. Along the way, some potential pitfalls for the numerical implementation of the decomposition method are identified and discussed. We conclude that the exact, nonlinear decomposition method introduced is a powerful and promising tool for understanding the nature of the decomposition of MHD waves as well as analyzing and interpreting the output of dynamic MHD simulations.

Why bother decomposing MHD disturbances?

- Key to understanding the evolution of stars and fusion reactors
- MHD waves have long been deemed crucial in coronal heating, and potentially igniting reconnection
- MHD modes have different qualities with implications for wave-energy transport.
- Intricately coupled and extremely complicated to tease out
- We often need decomposition to analyze data

The need for an exact decomposition method

All previous decomposition methods

- Are inexact
- Are only vaguely reliable in extreme plasma-beta regimes
- Use only a part of the available information
- Are extremely laborious and time-consuming

Long story short, we need a new plan!

MHD Equations

$$
\begin{cases}
\partial_t \rho + \nabla.(\rho \boldsymbol{v}) = 0 \\
\partial_t \boldsymbol{v} + \boldsymbol{v}.\nabla \boldsymbol{v} + \dfrac{1}{\rho}(\nabla p + \boldsymbol{j} \times \boldsymbol{B}) = 0 \\
\partial_t \boldsymbol{B} - \boldsymbol{B}.\nabla \boldsymbol{v} - \boldsymbol{B}\nabla.\boldsymbol{v} - \boldsymbol{v}.\nabla \boldsymbol{B} = 0 \\
\partial_t p + \boldsymbol{v}.\nabla p + \gamma p \nabla.\boldsymbol{v} = 0
\end{cases}
\Longleftrightarrow
$$

$$\dot{\boldsymbol{P}} + M_q \partial_q \boldsymbol{P} = 0$$

$$\boldsymbol{P} = (\rho, \boldsymbol{v}, \boldsymbol{B}, p)$$

Quasi-linear representation

The eigensystem of ideal-MHD equations

$$\mathrm{M}_q = R_q \Lambda_q L_q$$

$$\Lambda_q = \mathrm{diag}(v_q, v_q, v_q - a_q, v_q + a_q, v_q - c_{\mathrm{s},q}, v_q + c_{\mathrm{s},q}, v_q - c_{\mathrm{f},q}, v_q + c_{\mathrm{f},q})$$

Field divergence Entropy Alfvén Slow Fast

$$a_q = \frac{B_q}{\sqrt{\mu_0 \rho}} \; ; \qquad c_{\mathrm{f/s},q} = \frac{\sqrt{a^2 + c^2 \pm \sqrt{a^4 + c^4 + c^2\left(a^2 - 2a_q^2\right)}}}{\sqrt{2}}$$

The Eigenenergy Decomposition Method (EEDM)

$$\dot{\boldsymbol{P}} + M_q \partial_q \boldsymbol{P} = 0$$

$$\dot{\boldsymbol{P}} + R_q \boldsymbol{\mathcal{L}}_q(\boldsymbol{x}, \boldsymbol{\nabla}, t) = 0$$

$$\boldsymbol{\mathcal{L}}_q(\boldsymbol{x}, \boldsymbol{\nabla}, t) = L_q \Lambda_q \partial_q \boldsymbol{P}$$

The Eigenenergy Decomposition Method (EEDM)

$$\mathrm{E}_{\mathrm{tot}} = \tfrac{1}{2}\rho v^2 + \frac{p}{\gamma-1} + \frac{1}{\mu_0}\boldsymbol{B}\cdot\boldsymbol{B}$$

$$\dot{E}_{\mathrm{tot}} = \frac{1}{2}\dot{\rho}v^2 + \rho\dot{\boldsymbol{v}}\cdot\boldsymbol{v} + \frac{\dot{p}}{\gamma-1} + \frac{1}{\mu_0}\dot{\boldsymbol{B}}\cdot\boldsymbol{B}$$

$$= \sum_{m=1}^{8} \boldsymbol{c}_m(\boldsymbol{x},t)\cdot\overline{\mathcal{L}}_m(\boldsymbol{x},\boldsymbol{\nabla},t)$$

$$\dot{E}_{\mathrm{tot}} = \sum_{q\in(x,y,z)} \left(\dot{E}_{\mathrm{div},q} + \dot{E}_{\mathrm{ent},q} + \dot{E}^{-}_{\mathrm{A},q} + \dot{E}^{+}_{\mathrm{A},q} + \dot{E}^{-}_{\mathrm{s},q} + \dot{E}^{+}_{\mathrm{s},q} + \dot{E}^{-}_{\mathrm{f},q} + \dot{E}^{+}_{\mathrm{f},q}\right)$$

Evolution of the Eigenenergies

$$\dot{E}_{\mathrm{div},q} = -\frac{B_q \partial_q B_q}{\mu_0} v_q,$$

$$\mathcal{B}_{q',q} = (1 - \delta_{q,q'}) \begin{cases} a_{q'}/a_{\perp q}; & a_{\perp q} \neq 0 \\ 1/\sqrt{2}; & \text{otherwise} \end{cases}$$

$$\dot{E}_{\mathrm{ent},q} = -\frac{v^2 \left(c^2 \partial_q \rho - \partial_q p\right)}{2c^2} v_q,$$

$$\dot{E}^{\mp}_{\mathrm{A},q} = -\left(\sqrt{\mu_0 \rho} \boldsymbol{v} \times \boldsymbol{\mathcal{B}}_q\right)_q \left(\left(\sqrt{\mu_0 \rho} \partial_q \boldsymbol{v} \pm s_q \partial_q \boldsymbol{B}\right) \times \boldsymbol{\mathcal{B}}_q\right)_q \frac{\left(v_q \mp \|a_q\|\right)}{2\mu_0},$$

$$\dot{E}^{\mp}_{\mathrm{s/f},q} = -\left(\mathcal{S}_{\mathrm{s/f}} \alpha_{\mathrm{f/s},q} \left(\pm s_q c_{\mathrm{f/s},q} \sqrt{\mu_0 \rho} \boldsymbol{v} + c \boldsymbol{B}\right) \cdot \boldsymbol{\mathcal{B}}_q + \sqrt{\mu_0 \rho} \alpha_{\mathrm{s/f},q} \left(\pm c_{\mathrm{s/f},q} v_q - \frac{c^2}{\gamma - 1} - \frac{1}{2} v^2\right)\right)$$
$$\times \left(\mathcal{S}_{\mathrm{s/f}} \alpha_{\mathrm{f/s},q} \left(\pm s_q c_{\mathrm{f/s},q} \sqrt{\mu_0 \rho} \partial_q \boldsymbol{v} + c \partial_q \boldsymbol{B}\right) \cdot \boldsymbol{\mathcal{B}}_q + \sqrt{\mu_0 \rho} \alpha_{\mathrm{s/f},q} \left(\pm c_{\mathrm{s/f},q} \partial_q v_q - \frac{1}{\rho} \partial_q p\right)\right) \frac{\left(v_q \mp c_{\mathrm{s/f},q}\right)}{2\mu_0 c^2},$$

Raboonik et. al. 2024B (Equations 3)

Mode-decomposed energy

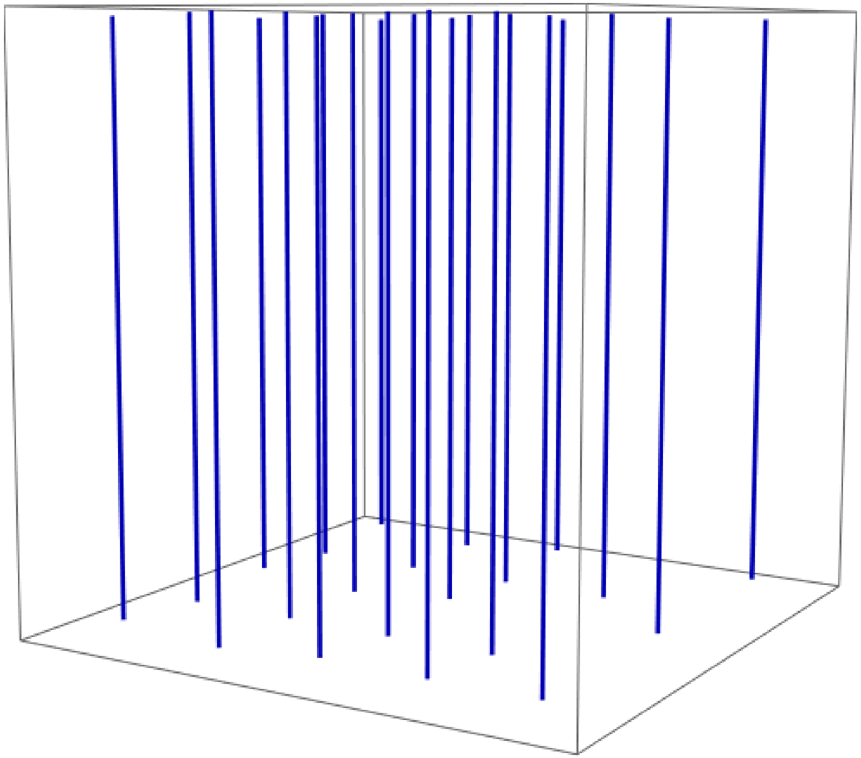
$$\underbrace{E_{\text{tot}} - E_0}_{\text{Pre-decomposition}} = \underbrace{\sum_{q\in(x,y,z)} \sum_{m=1}^{8} E_{m,q}}_{\text{Post-decomposition}}$$

$$\Delta E_{\text{tot}} = \Delta \mathcal{E}_{\text{tot}}$$

Eigenenergy initial conditions: $E_{m,q}(\boldsymbol{x}; t = t_0) = 0$

Simulation Ia:
High-beta case ($\beta = 9$)

Background field:

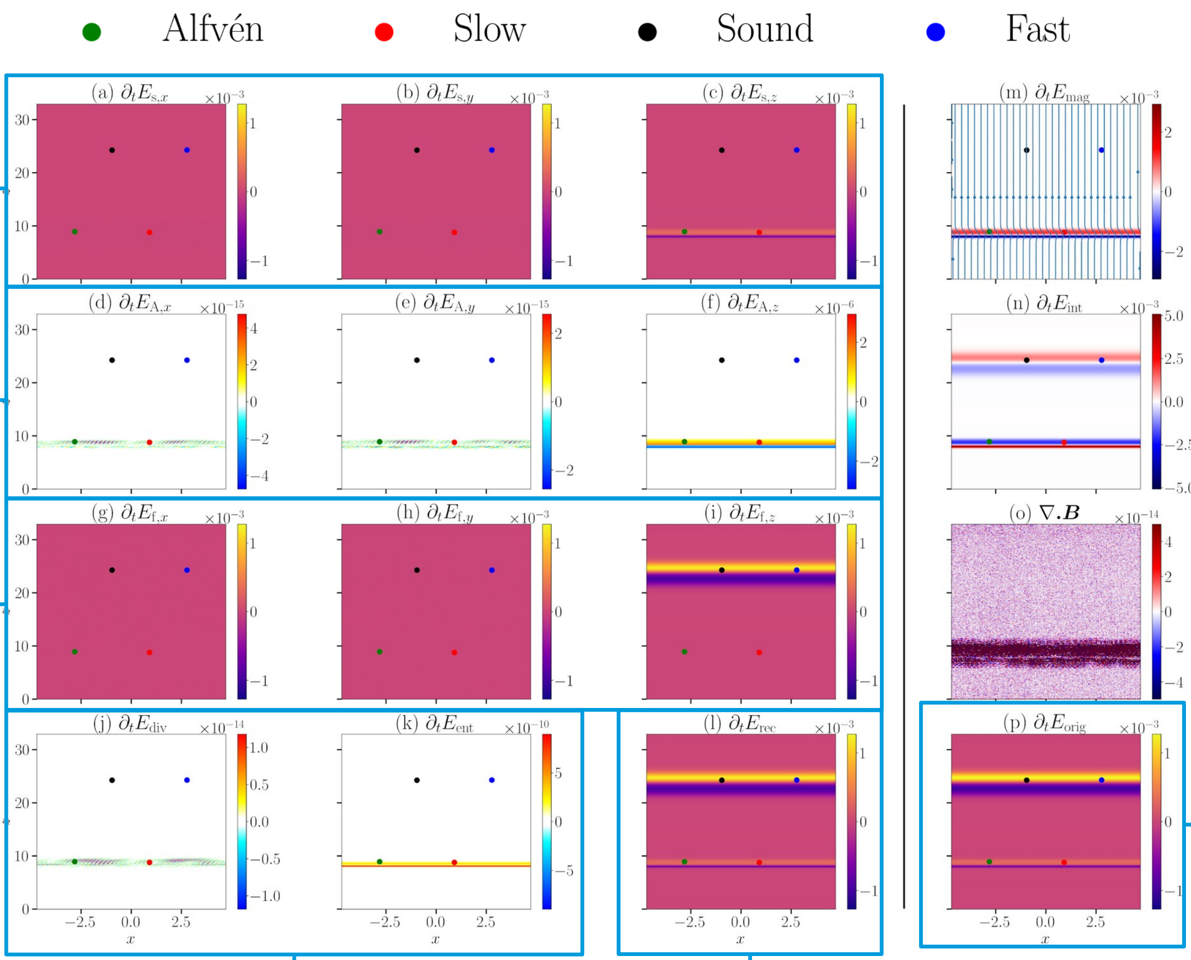


Driver: torsional Alfven wave

$\dot{E}^{-}_{\mathrm{s},q} + \dot{E}^{+}_{\mathrm{s},q}$

$\dot{E}^{-}_{\mathrm{A},q} + \dot{E}^{+}_{\mathrm{A},q}$

$\dot{E}^{-}_{\mathrm{f},q} + \dot{E}^{+}_{\mathrm{f},q}$



$\dot{E}_{\mathrm{tot}}$

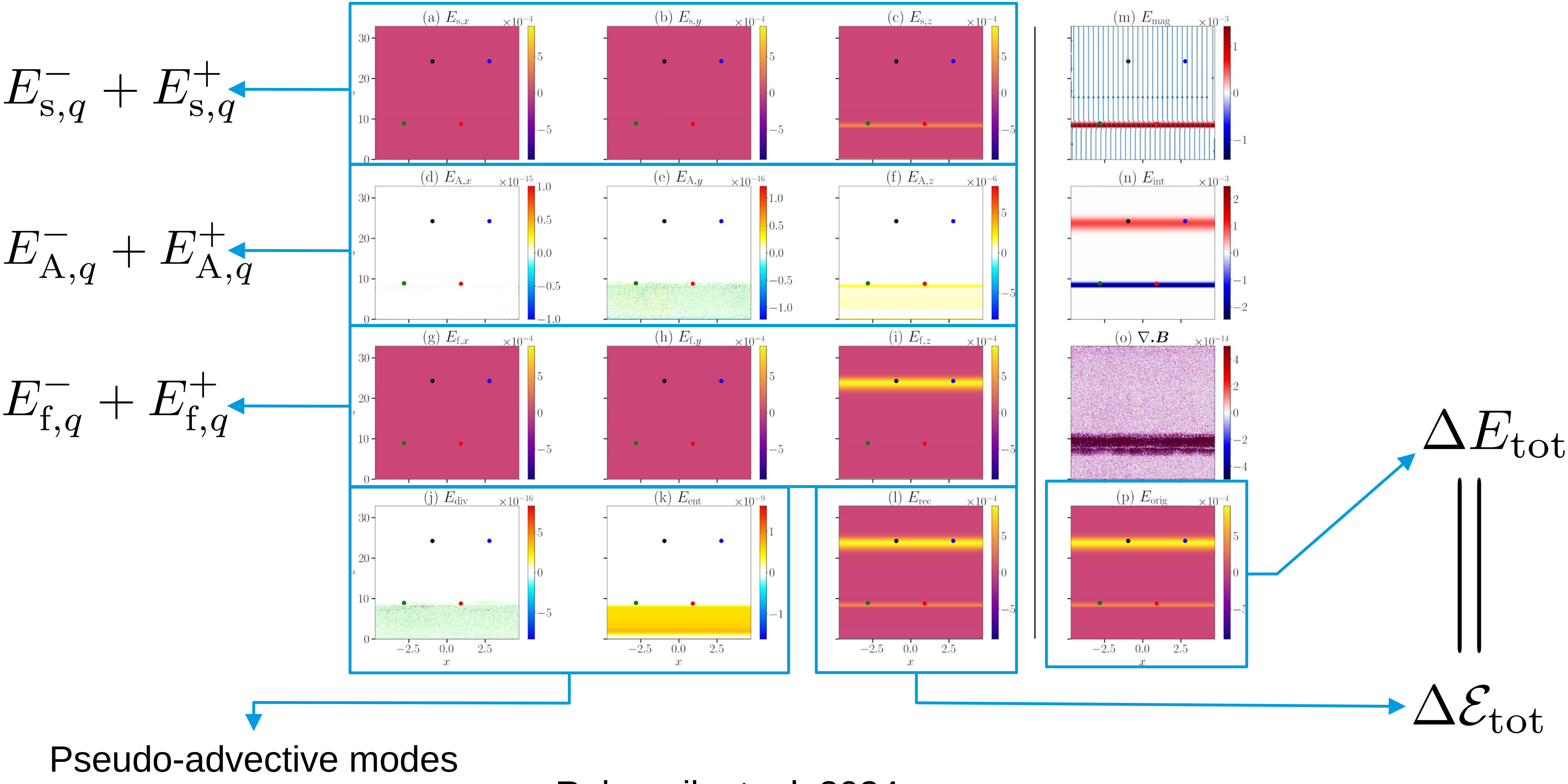
$\|$

$\dot{\mathcal{E}}_{\mathrm{tot}}$

Pseudo-advective modes

Raboonik et. al. 2024a

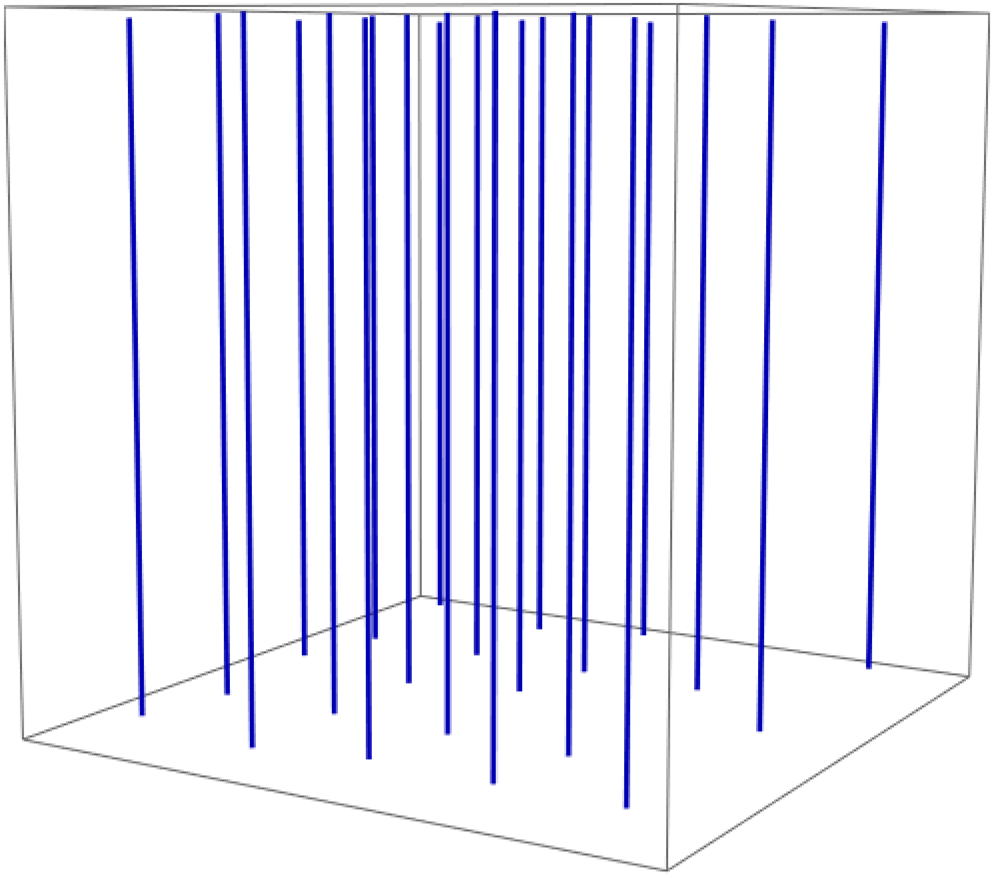
t = 17.95, y = -0.024

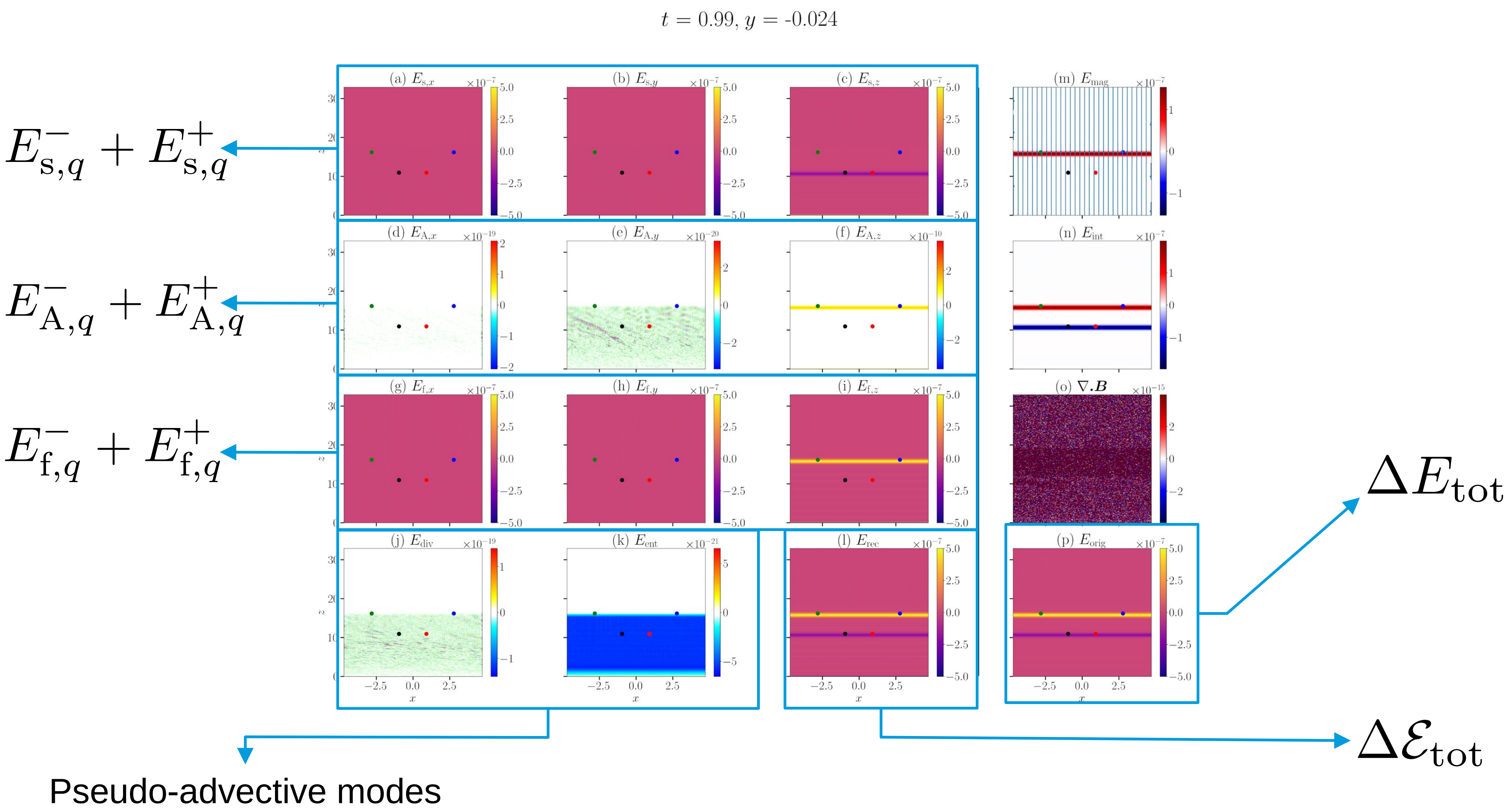


Raboonik et. al. 2024a

Simulation Ib: Low-beta case ($\beta = 0.54$)

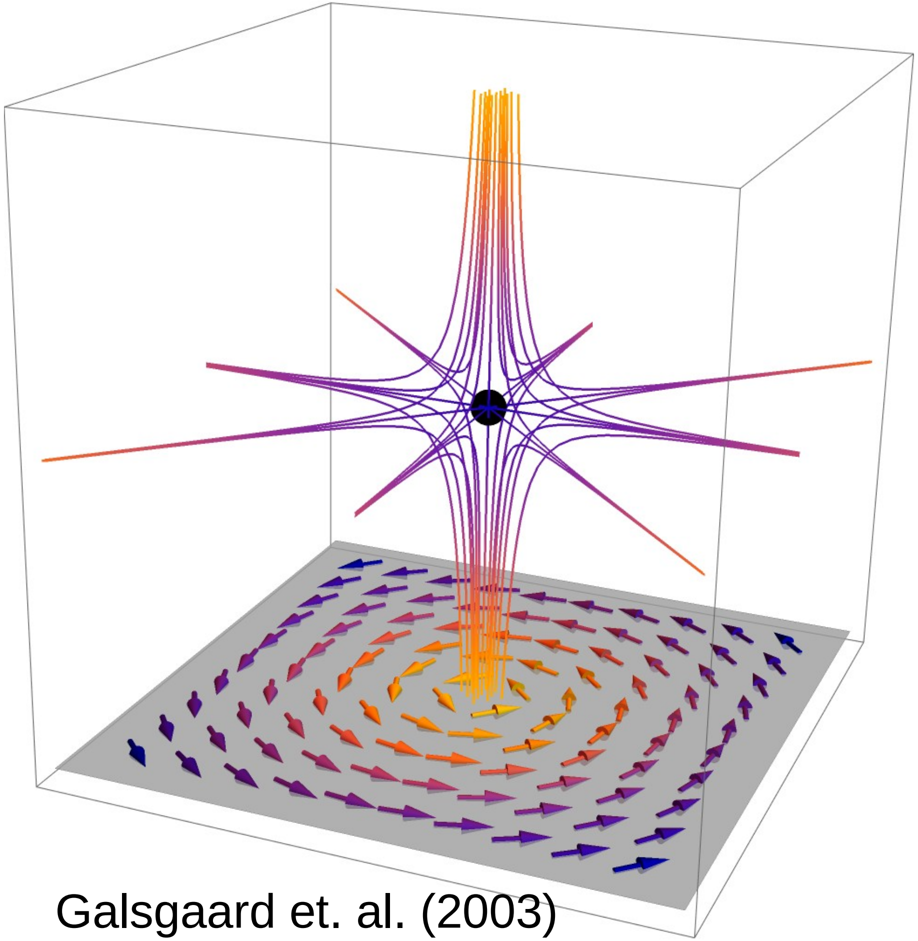
Background field





Raboonik et. al. 2024a

Simulation II:
3D wave-null interaction

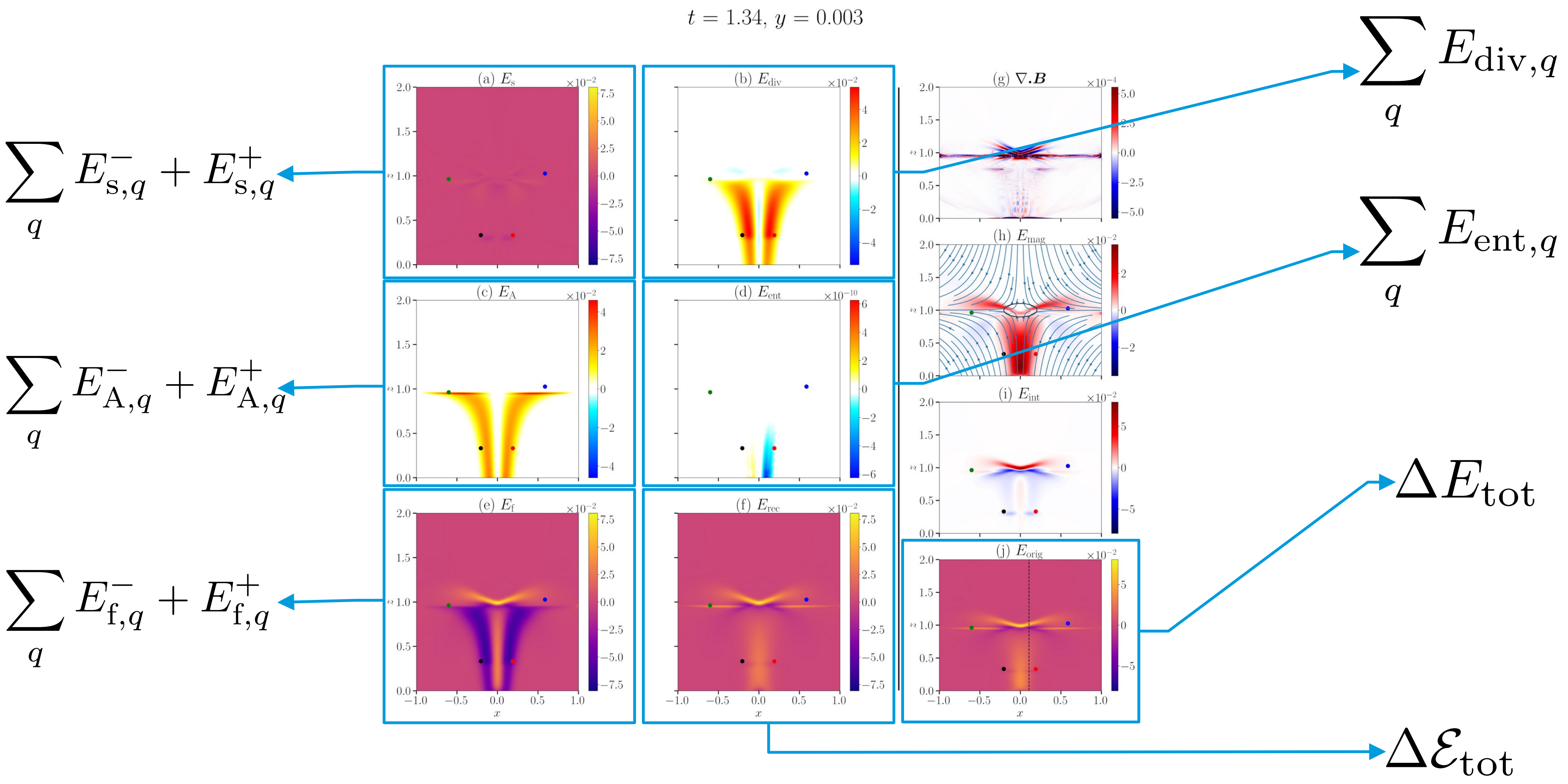


Galsgaard et. al. (2003)

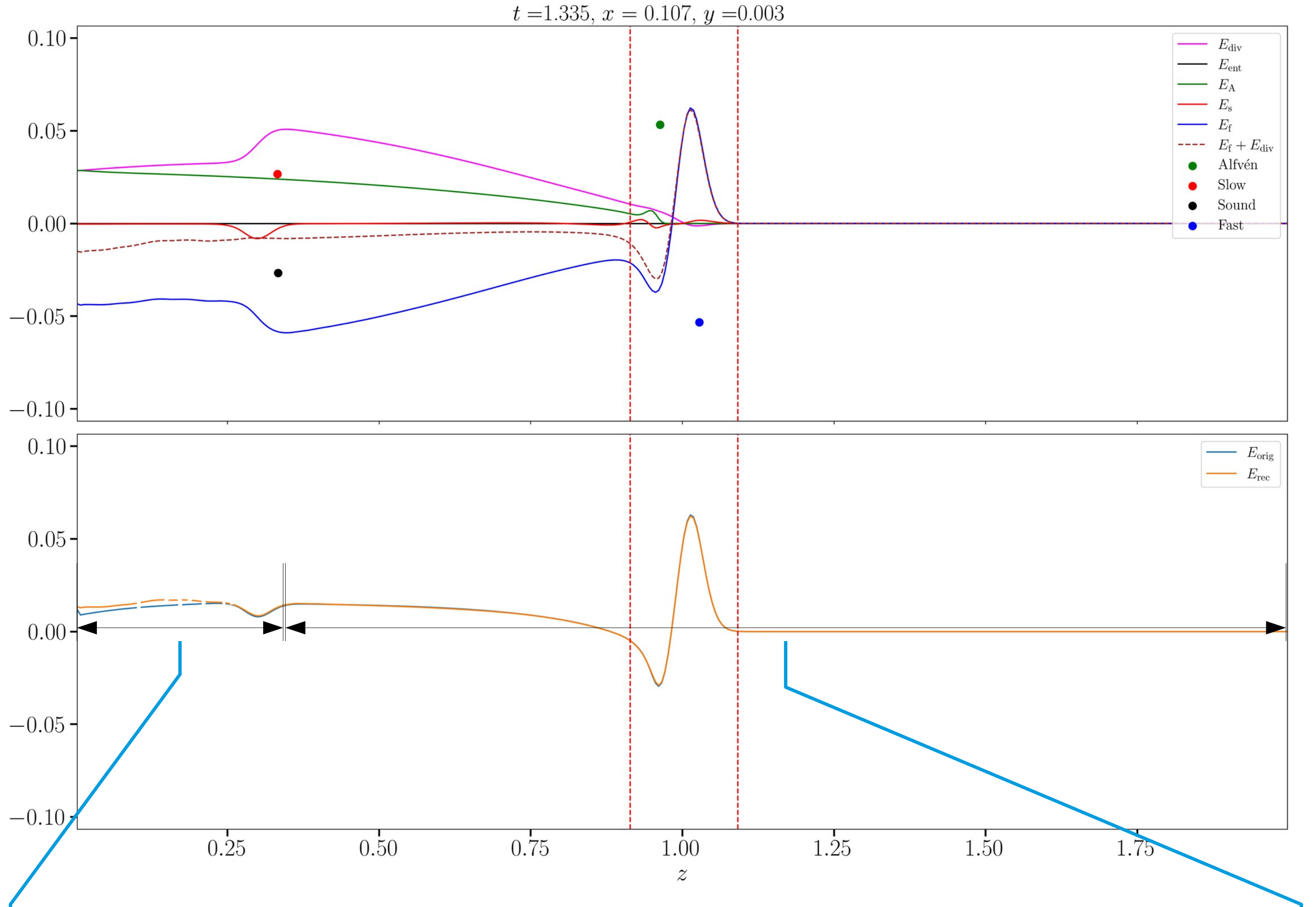
Driver:

$$\boldsymbol{v} = v_0 \exp\left(-\frac{x^2+y^2}{2\sigma_R^2}\right) \exp\left(-\frac{t-t_0}{2\sigma_{\mathrm{P}}^2}\right) (y, -x, 0)\,,$$

$$\boldsymbol{B} = \boldsymbol{B}_0 - \sqrt{\rho_0}\boldsymbol{v}$$



Raboonik et. al. 2024a



$\Delta E_{\mathrm{tot}} \approx \Delta \mathcal{E}_{\mathrm{tot}}$

Raboonik et. al. 2024a

$\Delta E_{\mathrm{tot}} = \Delta \mathcal{E}_{\mathrm{tot}}$

Conclusion

The EEDM

- Is exact and analytical
- Applies to linear and non-linear disturbances across the entire plasma-beta spectrum
- Offers a unique definition and identification of the ideal-MHD modes
- Uses all the available information on plasma evolution
- Offers precise energy measurements
- Detects mode conversion
- Is easy to apply both for analytical and simulation studies and the entire process can be automated for data analysis (stay tuned, the parallelized source code will be released on my GitHub)

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